DIFFUSION OF A DISSOLVED GAS IN A FLOW WITH A STATIONARY CAVITY

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An exact solution is obtained for the plane problem of the diffusion of a gas from a solution into a stationary cavity behind a symmetric profile for a zero cavitation number in an ideal liquid, and its approximate generalizations are indicated.

This paper considers the plane problem of the diffusion of a dissolved gas in the flow of an ideal liquid in the presence of a symmetric stationary cavity on the body in the streamlining flow. The liquid is assumed to be weightless, inviscid, and incompressible.

Separation of the gases dissolved in the liquid in the zones of reduced pressure [1–3] may exert considerable influence on the development of cavitational processes. The problem under consideration here may be of practical interest in view of the pronounced intensification of gas separation in the presence of a relative velocity at the phase boundary [4].

Under our assumptions the problem requires the joint consideration of the equations of motion for an ideal liquid and the Fick equation describing the diffusion of the dissolved substance in the flow. For the plane case in rectangular coordinates the Fick equation has the form

$$v_x \frac{\partial C}{\partial x} + v_y \frac{\partial C}{\partial y} = k\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right). \tag{1}$$

Here k is the coefficient of molecular diffusion; C is the concentration of the solution; v_x and v_y are the projections of the flow velocity, and these are functions of the coordinates x, y.

Since the concentration C is not included in the equations of motion, determination of the functions v_x and v_y for the cavitation streamlining regime for a body of the given shape involves a conventional hydrodynamic problem whose solution we assume to be available. Bearing in mind the constancy of pressure at the cavity boundary, we can write the boundary condition at the cavity for the diffusion problem in accordance with the Henry-Dalton law as

$$C = C_s = C_{sa}\frac{p_c}{p_a}. \tag{2}$$

Absence of gas exchange with the body at the boundary yields

$$\partial C/\partial n = 0. \tag{2'}$$

In the unperturbed flow

$$C = C_\infty. \tag{2''}$$

In boundary conditions (2), (2'), and (2") we have used the following notation: p_c and p_a are the pressure in the cavity and the atmospheric pressure, respectively; C_s and C_{sa} are the saturation concentrations corresponding to these pressures; n is the normal at the boundary of the body.

In boundary condition (2) we assume the equation for the cavity contour to be known from the solution of the hydrodynamic problem. It is possible to obtain a rigorous solution in general form of Eq. (1) with boundary conditions (2), (2'), and (2"), which requires no specification of v_x and v_y or the cavity contour, by turning to the hydrodynamic stream functions ψ and the potential φ, and taking the lines of constant ψ and φ rather than the x- and y-coordinate axes to be the coordinate lines. In this case, everywhere with the exception of the critical point at which $v_x^2 + v_y^2 = 0$, Eq. (1) transforms to the simplest form

$$\frac{1}{k}\frac{\partial C}{\partial \varphi} = \frac{\partial^2 C}{\partial \varphi^2} + \frac{\partial^2 C}{\partial \psi^2}. \tag{3}$$

The change of variables made it possible to eliminate the variable coefficients from the main equation. Significant also is the fact that unlike Eq. (1), Eq. (3) is constant for all possible shapes of the streamlined body and the cavity. The boundaries of the region for which a solution is being sought in the φ, ψ plane are also significantly simplified and assume the form of a simple section along the φ-axis for any shape of the streamlined body.

In this connection, the boundary conditions of the problem can be written in the form

$$C = C_\infty, \quad C = C_{sa}\frac{p_c}{p_a} \tag{4}$$

at an infinite distance from the body and at the boundary of the section, respectively.

The condition specifying an absence of gas exchange at the boundary of the body is satisfied automatically as a result of flow symmetry. The results thus obtained indicate that the possible quantitative variations in the conditions of the initial problem—associated with a specific body shape and the conditions of its streamlining gas defined by the cavitation number

$$\sigma = (p_\infty - p_c)\left(\frac{\rho v_\infty^2}{2}\right)^{-1},$$

for the diffusion problem—affect only the values of the velocity potential at the extreme points of the cavity. Here p_∞ and v_∞ are, respectively, the pressure and velocity in the unperturbed flow.

A rigorous solution of Eq. (3) for boundary conditions (4) can be obtained only for the case $\sigma = 0$, which corresponds to a cavity of infinite length, found in a regime of jet streamlining (Kirchhoff flow). Extension to the case of finite cavity dimensions may be achieved analogously to the case of friction of a viscous liquid on a plate of finite length, assuming that the neglected portion of the semi-infinite section has no effect on the diffusion processes at a point located upstream.

Considering the boundary conditions, it is possible to achieve a further simplification of the problem by turning to parabolic coordinates:

$$\varphi = \xi^2 - \eta^2, \quad \psi = 2\xi\eta,$$

as a result of which the partial differential equation (3) changes into an ordinary differential equation

$$\frac{d^2C}{d\eta^2} = -\frac{1}{k} 2\eta \frac{dC}{d\eta}. \tag{5}$$

The solution of Eq. (5) with respect to the boundary conditions for $\sigma = 0$ is expressed by the probability integral as a function of the complex argument

$$C(\varphi; \psi) = (C_\infty - C_s) \times$$

$$\times \Phi\left(\sqrt{\sqrt{\left(\frac{\varphi - \varphi_1}{k}\right)^2 + \psi^2} - \frac{\varphi - \varphi_1}{k}}\right) + C_s, \tag{6}$$

where φ_1 is the value of the potential at the forward point of the cavity. To calculate the values of the concentrations C from solution (6) at a given point on the (x, y)-plane it is still necessary to solve the hydrodynamic problem to determine the potential and the stream function in Eq. (6). For practical purposes, the determination of the gas flow rate through the cavity boundary is of great interest. The specific features of cavitating flows make it possible for this problem to achieve the transition to physical space in the general form.

If S denotes the length of the cavity boundary from its point of convergence with the body, n the normal to the cavity boundary (with the normal directed into the flow), and v_c the velocity at the cavity boundary, we can write the familiar relationships:

$$\frac{\partial C}{\partial n} = \frac{\partial C}{\partial \psi} v_c, \tag{7}$$

$$\frac{\partial \varphi}{\partial s} = v_c, \tag{8}$$

$$v_c = v_\infty \sqrt{1+\sigma}. \tag{9}$$

Having integrated (8) along the cavity boundary, because v_c = const, we obtain

$$\varphi - \varphi_1 = v_c s. \tag{10}$$

Having differentiated (6) with respect to ψ with consideration given to (7), (9), and (10), for the concentration gradient along the normal to the cavity boundary $\psi = 0$ we obtain the expression

$$\frac{\partial C}{\partial n} = \frac{C_\infty - C_s}{\sqrt{\pi}} \sqrt{\frac{v_\infty}{ks}} \sqrt[4]{1+\sigma}. \tag{11}$$

The per-second mass flow rate of the gas into the cavity through the element dS of the cavity boundary according to the Nernst law is equal to

$$dM = \frac{\partial C}{\partial n} ds. \tag{12}$$

After integration of (12) with consideration of (11) for the total gas flow rate into a symmetric cavity with two boundaries each of which exhibits a length L along the curve and a width B along the generatrix, it is possible to derive the formula

$$M = 4k \frac{C_\infty - C_s}{\sqrt{\pi}} \sqrt[4]{1+\sigma} \sqrt{\frac{vL}{k}} B. \tag{13}$$

Analysis of formula (13) shows that in approximate terms it can be derived if the cavity is assumed to be thin and it is replaced by its section in the physical plane along the x-axis, and if it is assumed in the velocity projections that

$$v_x \approx v_c; \quad v_y \approx 0.$$

The additional assumption of the smallness of $\partial^2 C/\partial x^2$ in comparison with $\partial^2 C/\partial y^2$ makes it possible to extend formula (13) to the axisymmetric problem.

NOTATION

C is the dissolved gas concentration; k is the gas diffusivity coefficient in a solution; x and y are the rectangular coordinates; ξ and η are the parabolic coordinates; φ and ψ are the potential and function of current; v is the flow velocity; p is the pressure; ρ is the liquid density; σ is the cavitation number; n is the normal to the boundary of a cavity; s is the arc length along the cavity boundary; M is the gas mass released into a cavity per unit time; L is the cavity length along its boundary; B is the width or size of a body and a cavity.

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